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On Asymptotics of L_p Extremal Polynomials on a Complex Curve $(0<p<\infty)$

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Asymptotics of $L_p(\sigma)$ extremal polynomials on a closed curve are investigated. The measure σ is supposed to be concentrated on a Jordan curve in the complex plane and has masses in the exterior of the curve. The case $0<p<\infty$ is discussed in detail. © 1993 Academic Press, Inc.

INTRODUCTION

Let F be a compact set in the complex plane, and B is a metric space of functions, defined on F. Suppose B contains all the polynomials. The extremal or general Tchebyscheff n-polynomial is a monic nth degree polynomial T_n with a minimal distance from zero in B:

$$m_n(B)=\|T_n\|:=\inf\{\text{dist}(z^n+a_{n-1}z^{n-1}+\cdots+a_0, 0); a_{n-1}, ..., a_0\in C\}.$$

If B is the space of continuous functions on F with the supremum norm, then T_n is a usual Tchebyscheff polynomial and m_n is a Tchebyscheff constant associated with the compact F.

There are many interesting problems about extremal polynomials. The most important ones are their asymptotic and zero distributions. Recently, a series of results was established for the case of $L_p(F,\sigma)$ spaces ($1\leqslant p\leqslant\infty$, σ is a Borelean measure on F). The case $p=2$ is the special case of σ-orthogonal polynomials; it has a long history of study and will not be described here (see, for example, the classical Szegő [16] and Nevai [11–13] books and also the fundamental Widom paper [19]). If we are interested in so-called power asymptotics of extremal constants and polynomials, then the cases studied are the following:

(a) $F=[-1,1]$, $d\sigma(x)=\rho(x)\,dx$ (ρ is a weight function), $\rho(x)$ is non-negative and integrable. For $p=\infty$, $\rho(x)\equiv 1$ we have the classical Tchebyscheff polynomials.

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For $1 \leqslant p \leqslant \infty$, $\rho(x) = t(x)/\sqrt{1-x^2}$ and $\log t(x)$ a Riemann integrable function, Bernstein [1] found the power asymptotic of the extremal constants $m_{n,p}(\rho)$. The important generalization of this result was obtained by Lubinsky and Saff [10]. They proved the asymptotic of $m_{n,p}(\rho)$ and $T_{n,p}$ (outside the interval $[-1, 1]$) under a much more general condition on a weight function: $1/\rho(x) \in L_r[-1, 1]$, $\forall r > 1$ (Nevai condition).

(b) F is a closed rectifiable Jordan curve with some condition of smoothness.

The case $0 < p < +\infty$ was studied by Geronimus [3]. He proved power asymptotic of $m_{n,p}(\rho)$ and power asymptotic of $T_{n,p}$ on and outside the curve (with an additional condition that absolutely continuous part of measure σ satisfy the Szegő condition, see later). The case $p = \infty$ was investigated by Widom [19] for $F = \bigcup E_k$, E_k being a smooth closed Jordan curve. Recently, Li and Pan [9] studied the zero distributions of $L_p(\sigma)$ extremal polynomials on the unit circle $(1 < p < \infty)$.

(c) F is a rectifiable arc in the complex plane. This case is little understood. We do not know, for example, the power asymptotic of the weighted Tchebyscheff constants $m_{n,p}(\rho)$ $(0 < p \leqslant +\infty)$. The case $p = \infty$ is quite different from the classical real case (see [14, 18]). For F an arc on the unit circle, $p = \infty$, $\rho(x) = 1$ the Tchebyscheff polynomials were calculated in a terms of elliptic functions by Tiran and Detaille [17].

In this paper we shall study the power asymptotic of $m_{n,p}(\sigma)$ and $T_{n,p}$ in the case where $0 < p < \infty$, $F = E \cup \{z_1, z_2, ..., z_N\}$, E being a closed rectifiable Jordan curve with some smoothness condition, $z_k \in \Omega := \mathrm{Ext}(E)$, measure σ is a sum $\sigma = \alpha + \gamma$, with $\operatorname{supp} \alpha = E$, $\alpha' = \rho(\xi)$ on E, and γ is a discrete measure having a masses A_k in the points z_k. The result is the power asymptotic

$$T_{n,p}(z) = c(E)^n \Phi^n(z) \psi^*(z)[1 + \varepsilon_n(z)],$$

where $\varepsilon_n(z) \to 0$ uniformly on a compact subset of Ω, ψ^* is the solution of some extremal problem in the space $H_p(\Omega, \rho)$ $(0 < p < \infty)$. For $p = 2$ this result was obtained in our previous paper [5]; here we apply the same techniques for the general case and give a precise detials for $0 < p < 1$. In the first section we present a basic definition and lemmas in the $H_p(\Omega, \rho)$ spaces. In the second section, we prove the main result, Theorem 2.2.

1. Basic Definitions and Functional Spaces

1.1. *Conformal Mapping*

Let E be a Jordan closed rectifiable curve, $\Omega = \mathrm{Ext}(E)$, $G = \{w \in C, |w| > 1\}$. We note by $w = \Phi(z)$ the function that maps Ω conformally on G in such a manner that $\lim \Phi(z)/z > 0$ for $z \to \infty$ and $\Phi(\infty) = \infty$. Really, this limit is equal to $1/c(E)$, where $c(E)$ is the logarithmic capacity of E. Let Ψ be the inverse function to Φ, $\Psi: G \to \Omega$. The two functions $\Phi(z)$ and $\Psi(w)$ have a continuous extension to E and on the unit circle, respectively (Caratheodory Theorem [4]). Their derivatives $\Phi'(z)$ and $\Psi'(w)$ have no zeros in Ω and G and have limit values on E and on the unit circle almost everywhere (with respect to the Lebesgue measure). So the functions $\Phi'(z)$ and $\Psi'(w)$ are defined and integrable on E and on the unit circle. This gives us the possibility to define the analytic functions $(\Phi'(z))^{1/p}$ and $(\Psi'(w))^{1/p}$ for all $p: 0 < p < \infty$.

1.2. *H_p Spaces*

Let $\Delta = \{u \in C, |u| < 1\}$ be the unit disc. We start with the usual $H_p(\Delta)$ space. First suppose $1 \leqslant p < \infty$. One function $f(u) \in H(\Delta)$ (analytic in Δ) is from $H_p(\Delta)$ space if

$$\| f \|_{H_p}^p := \sup \int_0^{2\pi} |f(re^{i\Theta})|^p \, d\theta < \infty \ (0 < r < 1). \tag{1.1}$$

In this case f has limit values on the unit circle (almost everywhere) and the limit function is from the L_p class. Although we have

$$\int_0^{2\pi} |f(re^{i\Theta})|^p \, d\theta \leqslant \| f \|_{H_p}^p = \int_0^{2\pi} |f(e^{i\Theta})|^p \, d\theta,$$

this is the same as

$$\int_{|u| = r} |f(u)|^p \, |du| \leqslant r \int_{|u| = 1} |f(u)|^p \, |du|.$$

Now we define $H_p(G)$ as the space of functions $f(w)$ with $g(u) = f(1/u) \in H_p(\Delta)$. $H_p(G)$ is a Banach space. Each function $f(w)$ from this space is analytic in G, has limit values on the unit circle a.e. (see [15, 2, 4]), and for $1 < R$ we have

$$\int_{|w| = R} |f(w)|^p \, |dw| \leqslant R \int_{|w| = 1} |f(w)|^p \, |dw| = R \, \| f \|_{H_p(G)}^p.$$

For $0 < p < 1$, $H_p(\Delta)$ is not a normed space, but it is a metric space with the distance $d(f, g) = \| f - g \|_{H_p(\Delta)}^p$ ((1.1) as definition of $\|*\|$) and it is a complete space. Each function $f(w)$ of $H_p(G)$ has a decomposition $f = B(w)[h(w)]^{2/p}$, where $B(w)$ is the Blaschke product associated with zeros of $f(w)$ and $h(w) \in H_2(G)$ (see [15]). So the function $|f(w)|^p$ has limit values on the unit circle.

1.3. *Szegő Function*

Suppose σ a Borel measure on E with the absolutely continuous part $\rho(\zeta) = d\sigma/|d\zeta|$, $\zeta \in E$. The Szegő function $D(z)$ associated with the curve E and the weight function $\rho(\zeta)$ is the function defined by the following properties:

(i) $D(z)$ is analytic in Ω, $D(z) \neq 0$ in Ω, and $D(\infty) > 0$

(ii) $D(z)$ has limit values on E (a.e.) and

$$|D(\zeta)|^{-p} |\Phi'(\zeta)| = \rho(\zeta), \qquad \zeta \in E \text{ (a.e. on } E).$$

A sufficient condition for the existence of the Szegő function is the well known Szegő condition:

$$\int_E (\log \rho(\zeta)) \, |\Phi'(\zeta)| \, |d\zeta| > -\infty. \tag{1.2}$$

Under this condition we can get the Szegő function easily: first we define the weight function $\delta(w)$ on the unit circle by

$$\delta(e^{i\theta}) = \rho(\zeta)/|\Phi'(\zeta)|, \qquad \zeta = \Psi(e^{i\theta}).$$

Then $|\Phi'(\zeta)| \; |d\zeta| = d\theta$ and (1.2) imply the usual Szegő condition $\int_0^{2\pi} \log \delta(e^{i\theta}) \, d\theta > -\infty$.

The following function is the Szegő function for the domain $G = \{|w| > 1\}$:

$$D_G(w) = \exp\left\{ -\frac{1}{2p\pi} \int_0^{2\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log \delta(e^{i\theta}) \, d\theta \right\}$$

(see [16]). Then the function $D(z) = D_G(\Phi(z))$ is exactly the function satisfying (i)–(ii).

1.4. $H_p(\Omega, \rho)$ *Spaces* $(0 < p < +\infty)$

We say that a function $f(z)$ analytic in Ω is from $H_p(\Omega, \rho)$ space iff $f(\Psi(w))/D(\Psi(w))$ is a function from $H_p(G)$. $H_p(\Omega, \rho)$ is a Banach space $(1 < p < +\infty)$. Each function $f(z)$ from $H_p(\Omega, \rho)$ has limit values on E and

$$\| f \|^p_{H_p(\Omega, \rho)} = \int_E |f(\zeta)|^p \rho(\zeta) \, |d\zeta| = \lim \frac{1}{R} \int_{E_R} \frac{|f(z)|^p}{|D(z)|^p} |\Phi'(z) \, dz| \; (R \to 1), \tag{1.3}$$

where $1 < R$, $E_R = \{z \in \Omega\colon |\Phi(z)| = R\}$. For $0 < p < 1$, $H_p(\Omega, \rho)$ as above is a metric space with the quasi-norm (i.e., $\|\alpha f\|^p = |\alpha|^p \|f\|^p$ and $\|f + g\|^p \leqslant \|f\|^p + \|g\|^p$)

$$\|f\|^p_{H_p(\Omega,\rho)} = \sup \frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D(z)|^p} |\Phi'(z)\, dz|$$

$$= \lim \frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D(z)|^p} |\Phi'(z)\, dz| \; (R \to 1). \tag{1.4}$$

LEMMA 1.1. *If $f(z) \in H_p(\Omega, \rho)$ then for every compact set $K \subset \Omega$ there is a constant C_K such that*

$$\sup_K |f(z)| \leqslant C_K \|f\|^p_{H_p(\Omega,\rho)}.$$

Proof. The lemma follows from the Cauchy formula for $f(z)(\Phi'(z))^{1/P}/D(z)$ applied on the curve E_R $(1 \leqslant p)$ and Minkowsky inequality. For $0 < p < 1$, we note that function $|f(\Psi(w))/D(\Psi(w))|^p$ is subharmonic in G, and if $g(w)$ is a harmonic function with the same limit values on E_R one has $g(z) \geqslant |f(\Psi(z))/D(\Psi(z))|^p$, $z \in K$. The lemma follows from well known property of harmonic functions (representation by Poisson kernel).

LEMMA 1.2. *Let $\{f_n\}$ be a sequence of functions from $H_p(\Omega, \rho)$ and*

(i) *$f_n \to f$ uniformly on the compact sets of Ω*

(ii) *$\|f\|^p_{H_p(\Omega,\rho)} \leqslant M$ (constant).*

Then $f \in H_p(\Omega, \rho)$ and $\|f\|^p_{H_p(\Omega,\rho)} \leqslant \liminf \|f_n\|^p_{H_p(\Omega,\rho)}$.

Proof. The function $f(z)$ is analytic in Ω and

$$\frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D(z)|^p} |\Phi'(z)\, dz| = \lim \frac{1}{R}\int_{E_R} \frac{|f_n(z)|^p}{|D(z)|^p} \Phi'(z)\, dz| \leqslant M$$

imply that $f \in H_p(\Omega, \rho)$. Suppose $M^* = \liminf \|f_n\|^p_{H_p(\Omega,\rho)}$, then for $n \in \Lambda$ (subset of N) and $n > N_0$ we have $\|f_n\|^p_{H_p(\Omega,\rho)} \leqslant M^* + \varepsilon$. This implies

$$\frac{1}{R}\int_{E_R} \frac{|f(z)|^p}{|D(z)|^p} |\Phi'(z)\, dz| = \lim \frac{1}{R}\int_{E_R} \frac{|f_n(z)|^p}{|D(z)|^p} |\Phi'(z)\, dz| \leqslant M^* + \varepsilon.$$

Thus $\|f\|^p_{H_p(\Omega,\rho)} \leqslant M^* + \varepsilon$, $\forall \varepsilon > 0$. This proof is valid for all $p > 0$.

1.5. *Extremal Problems in the $H_p(\Omega, \rho)$ Spaces*

We pose $(0 < p < \infty)$

$$\mu(\rho) := \inf\{\|\phi\|^p_{H_p(\Omega,\rho)},\ \phi \in H_p(\Omega, \rho),\ \phi(\infty) = 1\}. \tag{1.5}$$

One can calculate the extremal function in (1.5) explicitly from the Szegő function: first we have

$$\frac{1}{R}\int_{E_R}\frac{|\phi(z)|^p}{|D(z)|^p}|\Phi'(z)\,dz| = \frac{1}{R}\int_{|w|=R}\frac{|\phi(\Psi(w))|^p}{|D(\Psi(w))|^p}|dw| \geqslant \frac{2\pi}{D(\infty)^p} \qquad (1.6)$$

because the function under the integral symbol is subharmonic in G. If $\phi^* = D(z)/D(\infty)$, then in (1.6) we have equality exactly. So $\phi^*(z)$ is an extremal function for (1.5).

LEMMA 1.3. *An extremal function* ψ^* *of the problem*

$$\mu^*(\rho) := \inf\{\|\phi\|^p_{H_p(\Omega,\rho)},\ \phi\in H_p(\Omega,\rho),\ \phi(\infty)=1,\ \phi(z_k)=0,\ k=1, ..., N\} \qquad (1.7)$$

is given by ($\phi^* = D(z)/D(\infty)$)

$$\psi^* = \phi^* \prod_{k=1}^{N} \frac{\Phi(z)-\Phi(z_k)}{\Phi(z)\,\bar{\Phi}(z_k)-1}\,\frac{|\Phi(z_k)|^2}{\Phi(z_k)} \qquad (z_k \in \Omega). \qquad (1.8)$$

Proof. We set

$$B(z) = \prod_{k=1}^{N} \frac{\Phi(z)-\Phi(z_k)}{\Phi(z)\,\bar{\Phi}(z_k)-1}\,\frac{|\Phi(z_k)|^2}{\Phi(z_k)},$$

$B(z)$ is a bounded analytic function in Ω, $B(\infty)=1$, $B(z)$ has a continuous extension to E, and $|B(\zeta)| = \prod_1^N |\Phi(z_k)|$ (Blaschke product). If $\phi\in H_p(\Omega,\rho)$ and $\phi(\infty)=1$, $\phi(z_k)=0$, then $f(z)=\phi(z)/B(z)\in H_p(\Omega,\rho)$ and $f(\infty)=1$. From the continuity of $B(z)$ on E one can find

$$\|f\|^p = \left(\prod_1^N |\Phi(z_k)|\right)^{-p} \|\phi\|^p,$$

which implies $\mu(\rho) \leqslant (\prod_1^N |\Phi(z_k)|)^{-p}\mu^*(\rho)$. Conversely, for $f\in H_p(\Omega,\rho)$, $f(\infty)=1$ the function $\phi(\zeta)=f(z)\,B(z)$ is from the same space and $\phi(\infty)=1$, $\phi(z_k)=0$. This implies $\mu^*(\rho)\leqslant(\prod_1^N |\Phi(z_k)|)^p\,\mu(\rho)$. So

$$\mu^*(\rho) = \left(\prod_1^N |\Phi(z_k)|\right)^p \mu(\rho) \qquad (1.9)$$

and the lemma follows.

1.6. *Closed Curves of the Class* Γ (Geronimus [3])

For a closed Jordan curve the Faber's polynomials $F_n(z)$ are defined by decomposition

$$\Phi^n(z) = F_n(z) + \lambda_n(z)$$

with $\lambda_n(z) = O(1/z)$ for $z \to \infty$. A curve E is said to be from the class Γ if

$$\lambda_n(\zeta) \to 0 \text{ uniformly on } E.$$

If $z = z(t)$ is a parametrization of the curve E (z: $[\alpha, \beta] \to E$, $z(\alpha) = z(\beta)$) then a sufficient condition for E to be in the class Γ is that $z'(t)$ is in a Lipschitz δ-class for some exponent δ. In this case $\lambda_n(\zeta) = O(1/n^{\delta'})$ with $0 < \delta' < \delta$ [8].

2. Asymptotics of Extremal Polynomials

Let E be a closed Jordan rectifiable curve, $\Omega := \text{Ext}(E)$, $z_1, ..., z_N \in \Omega$. Suppose that the measure σ is a sum $\sigma = \alpha + \gamma$ with $\text{supp}\, \alpha = E$, $d\alpha/|d\zeta| = \rho(\zeta)$ (absolutely continuous part of α), γ being a discrete measure with the masses A_k in the points z_k. We denote as in the introduction by $m_{n,p}(\sigma)$ the extremal constants: ($F = E \cup \{z_1, z_2, ..., z_N\}$)

$$m_{n,p}(\sigma) := \min\{\|Q_n(z)\|_{L_p(\sigma, F)}, Q_n(z) = z^n + \cdots\}$$

and by $T_{n,p}(z; \sigma)$ the associated extremal polynomials. First we state the result of Geronimus [3]:

Theorem 2.1. *If $0 < p < \infty$, E is from the class Γ, $\rho(\zeta)$ satisfy the Szegő condition* (1.2), *then*

(i) $\displaystyle \lim \frac{m_{n,p}(\alpha)}{c(E)^n} = (\mu(\rho))^{1/P}$

(ii) $\displaystyle \lim \left\| \frac{T_{n,p}(z; \alpha)}{c(E)^n \, \Phi^n(z)} - \phi^*(z) \right\|_{H_p(\Omega, \rho)} = 0$

(iii) $T_{n,p}(z; \sigma) = c(E)^n \, \Phi^n(z) \, \phi^*(z) \, [1 + \varepsilon_n(z)]$, $\varepsilon_n(z) \to 0$ *uniformly on the compact sets of Ω.*

The constant $\mu(\rho)$ and the function $\phi^*(z)$ are defined in 1.5 *of the previous section.*

More precisely, Geronimus proved that if E is from the class Γ, then (i), (ii), (iii), and Szegő condition (1.2) are equivalent to one another. Now we are able to prove

Theorem 2.2. *If $0 < p < \infty$, E is from the class Γ, $\rho(\zeta)$ satisfy the Szegö condition, then for a measure σ*

(i) $\lim_{n \to \infty} \frac{m_{n,p}(\sigma)}{c(E)^n} = (\mu^*(\rho))^{1/p}$

(ii) $\lim_{n \to \infty} \left\| \frac{T_{n,p}(z;\sigma)}{c(E)^n\, \Phi^n(z)} - \psi^*(z) \right\|_{H_p(\Omega,\rho)} = 0$

(iii) $T_{n,p}(z;\sigma) = c(E)^n\, \Phi^n(z)\, \psi^*(z)\, [1+\varepsilon_n(z)]$, $\varepsilon_n(z) \to 0$ *uniformly on compact subsets of* Ω.

The constant $\mu^*(\rho)$ *and the function* $\psi^*(z)$ *are defined in* 1.5 *of the previous section.*

We note that the form of asymptotic of $m_{n,p}(\sigma)$ and $T_{n,p}(z;\sigma)$ is the same as in the Geronimus theorem, only the extremal constant $\mu^*(\rho)$ and the extremal function $\psi^*(z)$ change. The asymptotics do not depend on a singular part of the measure σ (the same for α) on the curve E. The constants $\mu(\rho)$ and $\mu^*(\rho)$ and the functions $\phi^*(z)$ and $\psi^*(z)$ are given by (1.9) and (1.8).

Proof. (i) First we set $\phi_n(z) = Q_n(z)/c(E)^n\, \Phi^n(z)$ for a polynomial $Q_n(z) = z^n + \cdots$. Every function of this type is in the $H_p(\Omega, \rho)$ and $\phi_n(\infty) = 1$. Obviously, for this type of function

$$\|\phi_n\|^p_{H_p(\Omega,\rho)} = \int_E |\phi_n(\zeta)|^p\, \rho(\zeta)\, |d\zeta|. \tag{2.1}$$

We have

$$m_{n,p}(\sigma) = \min \left\{ \int_E |Q_n(\zeta)|^p\, d\alpha(\zeta) + \sum_{k=1}^{N} |Q_n(z_k)|^p\, A_k \right\}^{1/p}. \tag{2.2}$$

Suppose now that $Q_n(z) = Q_{n-N}(z)(z-z_1)(z-z_2)\cdots(z-z_N)$ where $Q_{n-N}(z) = z^{n-N} + \cdots$, then

$$m^p_{n,p}(\sigma) \leqslant \min \left\{ \int_E |Q_{n-N}(\zeta)|^p\, |\omega_N(\zeta)|^p\, d\alpha(\zeta) \right\},$$

where $\omega_N(z) = (z-z_1)(z-z_2)\cdots(z-z_N)$. The absolutely continuous part of the measure $|\omega_N(\zeta)|^p\, d\alpha(\zeta))$ is $|\omega_N(\zeta)|^p\, \rho(\zeta)\, |d\zeta|$, it satisfies the Szegő condition, and from Theorem 2.1 we get

$$\limsup \frac{m_{n,p}(\sigma)}{c(E)^n} \leqslant \left[\mu\left(\rho \frac{|\omega_N|^p}{c(E)^{N_p}} \right) \right]^{1/p}. \tag{2.3}$$

If now $\phi \in H_p(\Omega, \rho)$ and $\phi(\infty) = 1$, $\phi(z_k) = 0$, $k = 1, \ldots, N$, then

$\phi(z)\,c(E)^N\,\Phi^N(z)/\omega_N(z)$ is a function from $H_p(\Omega,\ \rho(|\omega_N|^p/c(E)^{N_p}))$ and for $\rho_N=\rho(|\omega_N|^p/c(E)^{N_p})$ we have

$$\|\phi\|_{H_p(\Omega,\,\rho)}=\|\phi c(E)^N\,\Phi^N/\omega_N\|_{H_p(\Omega,\,\rho_N)}\geqslant\mu\left(\rho\,\frac{|\omega_N|^p}{c(E)^{N_p}}\right)$$

(it is simple for $1\leqslant p<\infty$ because in this case $\|\phi\|^p_{H_p(\Omega,\,\rho)}=\int_E|\phi(\zeta)|^p\,\rho(\zeta)\,|d\zeta|$, for $0<p<1$ we use the continuity property of $\Phi(z)$ and (1.4)). Thus $\mu^*(\rho)\geqslant\mu(\rho(|\omega_N|^p/c(E)^{N_p}))$ (its are equal really). So we have from (2.3)

$$\limsup\frac{m_{n,p}(\sigma)}{c(E)^n}\leqslant[\mu^*(\rho))]^{1/p}. \tag{2.4}$$

Now (2.4) implies that $\|\phi_n^*\|_{H_p(\Omega,\,\rho)}\leqslant M=\text{const}$, $\phi_n^*=T_{n,p}(z;\sigma)/c(E)^n\,\Phi^n(z)$. Let $M^*:=\liminf\|\phi_n^*\|^p_{H_p(\Omega,\,\rho)}$, then for some subsequence $n\in\Lambda$, $M^*:=\lim\|\phi_n^*\|^p_{H_p(\Omega,\,\rho)}$. This and Lemma 1.1 imply that $\{\phi_n^*,\,n\in\Lambda\}$ is a normal family in Ω. So we can find a function $\psi(z)$ that is a uniform limit (on the compact subsets of Ω) of some subsequence $\{\phi_n^*,\,n\in\Lambda_1\}$ of $\{\phi_n^*,\,n\in\Lambda\}$. From Lemma 1.2, we get $\psi\in H_p(\Omega,\,\rho)$ and

$$\|\psi\|^p_{H_p(\Omega,\,\rho)}\leqslant\liminf\|\phi_n^*\|^p_{H_p(\Omega,\,\rho)}. \tag{2.5}$$

But, on the other hand, it is obvious that $\psi(\infty)=1$ and (2.4) implies that

$$\sum_{k=1}^N A_k\,|\Phi(z_k)|^{pn}\,|\phi_n^*(z_k)|^p\leqslant M=\text{const}.$$

That is, $|\phi_n^*(z_k)|=O(1/|\Phi(z_k)|^n)\to 0$ ($|\Phi(z_k)|>1$), we have finally $\psi(z_k)=0$, and from (2.5) we get

$$[\mu^*(\rho)]\leqslant\liminf\|\phi_n^*\|^p_{H_p(\Omega,\,\rho)}\leqslant\liminf\left[\frac{m_{n,p}(\sigma)}{c(E)^n}\right]^p.$$

This with (2.4) proves (i).

(ii) We set $\psi_n=\frac{1}{2}(\phi_n^*+\psi^*)$, then $\psi_n(\infty)=1$ and $\psi_n(z_k)\to 0$, $n\to\infty$ ($k=1,2,...,N$). As in (i), we get $\liminf\|\psi_n\|^p_{H_p(\Omega,\,\rho)}\geqslant\mu^*(\rho)$. Now (ii) follows from Clarkson inequality:

$1\leqslant p\leqslant 2$,

$$\left[\int_E|\tfrac{1}{2}(\phi_n^*+\psi^*)|^p\,\rho(\zeta)\,|d\zeta|\right]^{1/(p-1)}+\left[\int_E|\tfrac{1}{2}(\phi_n^*-\psi^*)|^p\,\rho(\zeta)\,|d\zeta|\right]^{1/(p-1)}$$
$$\leqslant\left[\tfrac{1}{2}\int_E|\phi_n^*|^p\,\rho(\zeta)\,|d\zeta|+\tfrac{1}{2}\int_E|\psi^*|^p\,\rho(\zeta)\,|d\zeta|\right]^{1/(p-1)};$$

$2 \leqslant p < \infty$,

$$\int_E |\tfrac{1}{2}(\phi_n^* + \psi^*)|^p \rho(\zeta)\, |d\zeta| + \int_E |\tfrac{1}{2}(\phi_n^* - \psi^*)|^p \rho(\zeta)\, |d\zeta|$$

$$\leqslant \tfrac{1}{2} \int_E |\phi_n^*|^p \rho(\zeta)\, |d\zeta| + \tfrac{1}{2} \int_E |\psi^*|^p \rho(\zeta)\, |d\zeta|$$

$0 < p < 1$. Once can use the Keldysh lemma (see [6]): If $f_n(u)$ is a sequence of analytic in the unit disc functions, $f_n \in H_p(\Delta)$, $f_n(e^{i\theta})$ is a limit values of f_n on the unit circle and $f_n(0) \to 1$ plus

$$\lim_{n \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f_n(e^{i\theta})|^p \, d\theta \right\} = 1,$$

then $\lim_{n \to \infty} (1/2\pi) \int_0^{2\pi} |f_n(e^{i\theta}) - 1|^p \, d\theta = 0$. We need a simple generalization of this statement:

LEMMA 2.1. *If $f_n(u)$ is a sequence of analytic functions $f_n \in H_p(\Delta)$, $f_n(e^{i\theta})$ is a limit values of f_n on the unit circle and $f_n(0) \to 1$, $f_n(u_k) \to 0$ $(k = 1, 2, ..., N$ $u_k \in \Delta)$ and*

$$\lim_{n \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f_n(e^{i\theta})|^p \, d\theta \right\} = \prod_{k=1}^{N} \frac{1}{|u_k|^p}.$$

Then

$$\lim_{n \to \infty} \int_0^{2\pi} |f_n(e^{i\theta}) - b(e^{i\theta})|^p \, d\theta = 0, \qquad \text{where} \quad b(u) = \prod_{k=1}^{N} \frac{u - u_k}{u\bar{u}_k - 1} \frac{\bar{u}_k}{|u_k|^2}.$$

Proof. First we note that $|b(e^{i\theta})|^p = \prod_{k=1}^N 1/|u_k|^p$ and $b(0) = 1$. For the functions $g_n(u) = f_n(u)/b(u)$ we have $\lim_{n \to \infty} \{(1/2\pi) \int_0^{2\pi} |g_n(e^{i\theta})|^p \, d\theta\} = 1$. On the other hand $g_n(u) = h_n(u) + \sum_{k=1}^N r_k f_n(u_k)/(u - u_k)$, where the constants r_k do not depend on n. But $f_n(u_k) \to 0$, so $\lim \{(1/2\pi) \int_0^{2\pi} |h_n(e^{i\theta})|^p \, d\theta\} = 1$ and $h_n(0) = g_n(0) - \sum_{k=1}^N r_k f_n(u_k)/u_k \to 1$. The functions h_n are analytic in the unit disc and we can apply the Keldysh lemma for this sequence of functions. Thus $\lim_{n \to \infty} \{(1/2\pi) \int_0^{2\pi} |h_n(e^{i\theta}) - 1|^p \, d\theta\} = 0$. This implies $\lim_{n \to \infty} \int_0^{2\pi} |h_n(e^{i\theta})\, b(e^{i\theta}) - b(e^{i\theta})|^p \, d\theta = 0$ and the lemma follows from this.

We get (ii) by applying Lemma 2.1 to the sequence $\phi_n^*(z)/\phi^*(z)$ with $u = 1/\Phi(z)$ (we recall that $\phi^*(z) = D(z)/D(\infty)$).

(iii) follows from (ii) and Lemma 1.1. Theorem 2.2 is proved.

The interesting question is the asymptotic of $L_p(\sigma)$-extremal polynomials in the case when $E = \bigcup E_l$, E_l being a closed rectifiable Jordan curves,

$l = 1, 2, ..., L$, and $F = E \cup \{z_1, z_2, ..., z_N\}$. We shall give a result in our future paper (this case needs some different techniques).

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